

Convergence of the Derivatives of Hermite–Fejér Interpolation Polynomials of Higher Order Based at the Zeros of Freud Polynomials

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We shall prove pointwise convergence of the derivatives of Hermite–Fejér interpolation polynomials of higher order based at the zeros of orthonormal polynomials with respect to Freud weights $\exp(-x^m)$, $m = 2, 4, 6, \dots$. © 1995 Academic Press, Inc.

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The purpose of this paper is to prove pointwise convergence of the derivatives of Hermite–Fejér interpolation polynomials of higher order based at the zeros of orthonormal polynomials with respect to a Freud weight of the form $\exp(-x^m)$ with an even positive integer m .

Let

$$Q(x) = \frac{1}{2}x^m, \quad w(x) = \exp(-Q(x)),$$

where $m = 2, 4, 6, \dots$. The orthonormal polynomials $p_n(w^2; x) = p_n(x) = \gamma_n x^n + \dots$, where $\gamma_n > 0$, are defined by the relation

$$\int_{-\infty}^{\infty} p_l(x) p_n(x) w^2(x) dx = \delta_{ln}.$$

These polynomials were investigated by Freud, e.g., [2, 3], and recently by many authors in connection with approximation theory. For detailed references and an extensive survey, readers may refer to Nevai [11].

We denote the zeros of $p_n(x)$ by x_{kn} , $k = 1, 2, \dots, n$, where

$$x_{1n} > x_{2n} > \dots > x_{nn}.$$

Let v be a positive integer, and let l be a non-negative integer such that $v - 1 \geq l$. For $f \in C^l(\mathbf{R})$, the Hermite-Fejér interpolation polynomial $L_n(l, v; f, x)$ of order (l, v) based at the zeros x_{1n}, \dots, x_{nn} is defined to be the unique algebraic polynomial of degree at most $vn - 1$ which satisfies

$$L_n(l, v; f, x_{kn}) = f(x_{kn}),$$

$$L_n'(l, v; f, x_{kn}) = f'(x_{kn}), \dots,$$

$$L_n^{(l)}(l, v; f, x_{kn}) = f^{(l)}(x_{kn}),$$

$$L_n^{(l+1)}(l, v; f, x_{kn}) = 0, \dots,$$

$$L_n^{(v-1)}(l, v; f, x_{kn}) = 0$$

for $k = 1, 2, \dots, n$. It is known that, for every $n = 1, 2, \dots$, $k = 1, 2, \dots, n$ and $r = 0, 1, \dots, v - 1$, there exists a unique polynomial $h_{rkn}(v; x)$ of degree $vn - 1$ satisfying

$$h_{rkn}^{(j)}(v; x_{pn}) = \delta_{rj} \delta_{kp}, \quad p = 1, 2, \dots, n, \quad j = 0, 1, \dots, v - 1$$

(cf. [8, Chap. I, Sect. 4]). The interpolation polynomial $L_n(l, v; f, x)$ is written in the form

$$L_n(l, v; f, x) = \sum_{k=1}^n \sum_{r=0}^l f^{(r)}(x_{kn}) h_{rkn}(v; x).$$

Since $L_n(l, v; f, x) = 1$ for $f(x) = 1$, we see that

$$\sum_{k=1}^n h_{0kn}(v; x) = 1.$$

We note that $L_n(0, 1; f, x)$ is the Lagrange interpolation polynomial based at the points x_{1n}, \dots, x_{nn} . We define the modulus of continuity of $f \in C(\mathbf{R})$ on an interval $[a, b]$ by $\omega([a, b]; f; h) = \sup\{|f(x) - f(y)|; |x - y| \leq h, x, y \in [a, b]\}$, $h > 0$.

Freud [4] and Nevai [9, 10] considered pointwise convergence of the Lagrange interpolation polynomials $L_n(0, 1; f, x)$ for the Hermite weight $\exp(-x^2)$, i.e., $m = 2$. Knopfmacher [6] estimated the rate of approximation of pointwise convergence of the polynomials $L_n(0, 1; f, x)$ for the class of

regular Freud weights which includes the weights $\exp(-x^m)$, $m = 2, 4, 6, \dots$. Recently, the authors [5] observed the behavior of pointwise convergence of Hermite-Fejér interpolation polynomials $L_n(0, v; f, x)$ of order $(0, v)$ for the weights $\exp(-x^m)$, $m = 2, 4, 6, \dots$, and showed that if v is even then for every continuous function $f(x)$, the sequence $\{L_n(0, v; f, x)\}$ converges uniformly to $f(x)$ on any compact interval, and showed that if v is odd then for every interval I , there exists a continuous function $f(x)$ such that $\limsup_{n \rightarrow \infty} \max_{x \in I} |L_n(0, v; f, x)| = \infty$. On the other hand, Balázs [1] treated convergence problems of the derivatives $L_n^{(j)}(0, 1; f, x)$ of Lagrange interpolation polynomials for $m = 2$, and proved that $|f^{(j)}(x) - L_n^{(j)}(0, 1; f, x)| \leq C\omega(\mathbf{R}; f^{(r)}; n^{-1/2}) n^{-r/2+j} \{\log n + \exp(x^2/2)\}$ for $|x| \leq x_{1n}$, $j = 0, \dots, r$. In this paper, we shall consider convergence problems of the derivatives $L_n^{(j)}(l, v; f, x)$ for arbitrary $v = 1, 2, \dots, 0 \leq l \leq v-1$ and $m = 2, 4, 6, \dots$.

Let q_n denote the unique positive solution of the equation $q_n Q'(q_n) = n$, that is,

$$q_n = \left(\frac{2n}{m}\right)^{1/m}.$$

Our theorem is as follows:

THEOREM. *Let v be a positive integer and let l be an integer such that $v-1 \geq l \geq 0$. Then, there exist positive constants c and K satisfying the following:*

(i) *The case $v-1 = l$: Let N be an integer such that $N \geq l$, and $f \in C^N(\mathbf{R})$. Then, for $|x| \leq cq_n$,*

$$\begin{aligned} |L_n^{(j)}(v-1, v; f, x) - f^{(j)}(x)| &\leq C(1 + |x|^{j(m-2)/2}) e^{vx^m/2} \\ &\quad \times \omega\left([-Kq_n, Kq_n]; f^{(N)}, \frac{q_n}{n}\right) \\ &\quad \times \left(\frac{q_n}{n}\right)^N n^j \log n \\ &\quad j = 0, 1, \dots, N \quad n = N+1, N+2, \dots \end{aligned}$$

(ii) *The case $v-1 > l$: Let $f \in C^l(\mathbf{R})$. Then, for $|x| \leq cq_n$,*

$$\begin{aligned} |L_n^{(j)}(l, v; f, x) - f^{(j)}(x)| &\leq C(1 + |x|^{j(m-2)/2}) e^{vx^m/2} \\ &\quad \times \omega\left([-Kq_n, Kq_n]; f^{(l)}, \frac{q_n}{n}\right) \\ &\quad \times \left(\frac{q_n}{n}\right)^l n^j \log n \\ &\quad j = 0, 1, \dots, l \quad n = l+1, l+2, \dots \end{aligned}$$

Here, C is a positive constant independent of n , x and f .

COROLLARY. (i) The case $v-1=l$: Let $N \geq l$. If $\lim_{h \rightarrow 0} \omega(\mathbf{R}; f^{(N)}; h) \log h = 0$, then for every $M > 0$,

$$\lim_{n \rightarrow \infty} \max_{|x| \leq M} |L_n^{(j)}(v-1, v; f, x) - f^{(j)}(x)| = 0$$

for $j=0, 1, \dots, [(1-1/m)N]$ (the integral part of $(1-1/m)N$).

(ii) The case $v-1 > l$: If $\lim_{h \rightarrow 0} \omega(\mathbf{R}; f^{(l)}; h) \log h = 0$, then for every $M > 0$,

$$\lim_{n \rightarrow \infty} \max_{|x| \leq M} |L_n^{(j)}(l, v; f, x) - f^{(j)}(x)| = 0$$

for $j=0, 1, \dots, [(1-1/m)l]$.

We remark that the condition $\lim_{h \rightarrow 0} \omega(\mathbf{R}; f^{(N)}; h) \log h = 0$ holds, e.g., if $f^{(N)} \in \text{Lip } \alpha$, $0 < \alpha \leq 1$. We mention that Balázs [1] has obtained the estimate mentioned above for $v=1$ and $m=2$.

For the proof of Theorem, we need a basic estimate given in the following:

PROPOSITION. Let $r=0, 1, \dots, v-1$. There exists a positive constant κ such that

$$\sum_{k=1}^n |h_{rkn}(v; x)| \leq C e^{v|x|/2} \left(\frac{q_n}{n}\right)^r \log n \quad (1.1)$$

for $x \in [-\kappa q_n, \kappa q_n]$ and $n=1, 2, \dots$, where C is a constant independent of x and n .

We remark that for $m=2$ and $v=1$ (and thus $r=0$), Freud [4, Theorem 1] has gotten the estimate $\sum_{k=1}^n |h_{0kn}(1; x)| \leq C \{\log n + \exp(x^2/2)\}$.

The proofs of Theorem and Proposition will be given in the next section. We summarize here some known results which are needed in the proofs.

(I) [6, Lemma 4.11]: (i) There exists a constant $K_1 > 0$ independent of n such that $x_{1n} \leq K_1 q_n$, $n=1, 2, \dots$.

(ii) There exist constants $C_1, C_2, \kappa_1 > 0$ independent of n and k such that $C_1 q_n/n < x_{k-1n} - x_{kn} < C_2 q_n/n$ for $x_{k-1n}, x_{kn} \in [-\kappa_1 q_n, \kappa_1 q_n]$.

Let $x_{(x,n)}$ denote the closest zero of $p_n(x)$ to x . If x is the midpoint of two zeros, then we define $x_{(x,n)}$ to be the closest zero of $p_n(x)$ on the left.

(II) [6, Theorem 3.7]: There exist constants $C_3, C_4, \kappa_2 > 0$ independent of n and x such that

$$C_3 |x - x_{(x,n)}| \frac{n}{q_n} q_n^{-1/2} \leq |p_n(x)| w(x) \leq C_4 |x - x_{(x,n)}| \frac{n}{q_n} q_n^{-1/2},$$

$$n = 1, 2, \dots \text{ for } x \text{ with } |x| \leq \kappa_2 q_n.$$

(III) Bernstein's inequality [12, 4.8(51)]: Let $\Delta_n(t) = n^{-1}(1-t^2)^{1/2} + n^{-2}$. Let $R_n(t)$ be a polynomial of degree n . Then, for $-1 \leq t \leq 1$ and $j = 0, 1, \dots$,

$$|R_n^{(j)}(t)| \leq C_5 \Delta_n(t)^{-j} \max_{|s| \leq 1} |R_n(s)|, \quad n = 1, 2, \dots,$$

where C_5 is a positive constant depending only on j .

(IV) [7, Corollary 1, Theorem 3]: Let $r = 0, 1, \dots$, and $g(t) \in C^r(\mathbf{R})$. Let $R_n^*(t)$ be the polynomial of best approximation of order n to $g(t)$ on the interval $[-1, 1]$. Then, for $|t| \leq 1$,

$$(i) \quad |g^{(j)}(t) - R_n^{*(j)}(t)| \leq C_6 n^{-r} \Delta_n(t)^{-j} E_{n-r}(g^{(r)}; [-1, 1]),$$

$$j = 0, 1, \dots, r, n = r+1, r+2.$$

$$(ii) \quad |R_n^{*(j)}(t)| \leq C_7 n^{-r} \Delta_n(t)^{-j} \omega\left([-1, 1]; g^{(r)}; \frac{1}{n}\right),$$

$$j = r+1, r+2, \dots, n = 1, 2, \dots,$$

where C_6 is a positive constant depending only on r and C_7 is a positive constant depending only on j , and $E_{n-r}(g^{(r)}; [-1, 1]) = \max_{|t| \leq 1} |g^{(r)}(t) - T_{n-r}^*(t)|$, where $T_{n-r}^*(t)$ is the polynomial of best approximation of degree $n-r$ to $g^{(r)}(t)$.

Throughout this paper, the letters $C_1 \sim C_6, K_1, \kappa_1, \kappa_2$ with subscript are always the constants in the properties (I) ~ (IV). For the rest of the paper, the letter C denotes a positive constant which may differ at each different occurrence, even in the same chain of inequalities.

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Let $l_{kn}(x)$, $k = 1, 2, \dots$ be the fundamental polynomial of Lagrange interpolation polynomial $L_n(0, 1; f, x)$, that is, $l_{kn}(x) = h_{0kn}(1; x)$. Then,

$$l_{kn}(x) = \frac{p_n(x)}{(x - x_{kn}) p'_n(x_{kn})}, \quad k = 1, 2, \dots, n. \quad (2.1)$$

We note that $h_{rkn}(v; x)$ is divided by $l_{kn}^v(x)$ ($= \{l_{kn}(x)\}^v$) and $x = x_{kn}$ is a root with multiplicity r of $h_{rkn}(v; x)$. We define $e_{ir}(v; k, n)$, $i = r, r+1, \dots, v-1$ to be the coefficients in the expression

$$h_{rkn}(v; x) = l_{kn}^v(x) \sum_{i=r}^{v-1} e_{ir}(v; k, n)(x - x_{kn})^i, \\ k = 1, 2, \dots, n. \quad (2.2)$$

After this, if there is no possibility of misunderstanding, we write briefly

$$x_k = x_{kn}; \quad L_n(x) = L_n(l, v; f, x); \quad h_{rk}(x) = h_{rkn}(v; x); \\ l_k(x) = l_{kn}(x); \quad e_{ir}(k) = e_{ir}(v; k, n); \quad \omega(h) = \omega([a, b]; f, h).$$

We first prove the Proposition. By (2.1) and (2.2), we have

$$\sum_{k=1}^n |h_{rk}(x)| \leq \sum_{i=r}^{v-1} \sum_{k=1}^n \left| \frac{p_n(x)}{(x - x_k) p'_n(x_k)} \right|^v |e_{ir}(k)| |x - x_k|^i \\ := \sum_{i=r}^{v-1} \sum_{k=1}^n R_k(i, r, n; x).$$

Our task is to estimate $\sum_{k=1}^n R_k(i, r, n; x)$. To do so, we shall divide the sum into three parts. Here, we need a lemma on the behavior of $p'_n(x)$ in a neighborhood of x_k .

LEMMA 1 [5, Lemma 1]. *There exist constants $\tilde{\delta} > 0$ and $\tilde{\kappa} > 0$ such that $k < n$, $x_k \in [-\tilde{\kappa}q_n, \tilde{\kappa}q_n]$ and $x_k - \tilde{\delta}q_n/n \leq x \leq x_k + \tilde{\delta}q_n/n$, then*

$$C \frac{n}{q_n} q_n^{-1/2} w(x_k)^{-1} \leq |p'_n(x)| \leq C \frac{n}{q_n} q_n^{-1/2} w(x_k)^{-1},$$

where C is independent of k, n and x .

By (I), we may suppose that the constant $\tilde{\kappa}$ satisfies $x_n < -\tilde{\kappa}q_n$ and $\tilde{\kappa}q_n < x_1$. Let κ be a positive constant such that $\kappa < \tilde{\kappa}$, and let $x \in [-\kappa q_n, \kappa q_n]$. We choose δ so that $0 < \delta < \min\{C_1/2, \tilde{\delta}\}$ where $\tilde{\delta}$ and C_1 are the constants in the lemma and in (I), (ii), respectively. Let

$$J = \{k; |x - x_k| < \delta q_n/n\}, \\ J(j) = \{k; j \delta q_n/n \leq |x - x_k| < (j+1) \delta q_n/n, |x_k| \leq \kappa q_n\}, \\ j = 1, 2, \dots, \\ I = \{k; \delta q_n/n \leq |x - x_k|, \kappa q_n < |x_k|\}.$$

The sets J , $J(j)$ and I may depend on x and n . The set J contains at most one element and each of the sets $J(j)$, $j=1, 2, \dots$ contains at most two elements, and $\{1, 2, \dots, n\} = \bigcup_{j=0}^{\lambda(n)} J(j) \cup J \cup I$, where $\lambda(n)$ is the smallest number exceeding $2K_1 n/\delta$. Here, K_1 is the constant in (I), (i). Let

$$\begin{aligned}\Sigma_1 &= \sum_{k \in J} R_k(i, r, n; x), & \Sigma_2 &= \sum_{j=1}^{\lambda(n)} \sum_{k \in J(j)} R_k(i, r, n; x), \\ \Sigma_3 &= \sum_{k \in I} R_k(i, r, n; x).\end{aligned}$$

Then, $\sum_{k=1}^n R_k(i, r, n; x) = \Sigma_1 + \Sigma_2 + \Sigma_3$. To estimate Σ_p , $p=1, 2, 3$, we need bounds of the coefficients $e_{ir}(k)$ in (2.2). We shall get the bounds by using the following estimate:

LEMMA 2 [5, Lemma 5]. *Let v be a positive integer, and let $s=0, 1, \dots$. Then,*

$$|\{l_k^v(x)\}^{(s)}|_{x=x_k} \leq CM_n(x_k)^{\langle s \rangle} q_n^{(s-\langle s \rangle)(m-1)}, \quad k=1, 2, \dots, n.$$

where $\langle s \rangle = 1$ (s : odd), $\langle s \rangle = 0$ (s : even), and $M_n(x_k) = \max\{|x_k| q_n^{-2}, |x_k|^{m-1}\}$, and C is independent of n and k .

LEMMA 3. *For $k=1, 2, \dots, n$ and $r=0, 1, \dots, v-1$,*

$$|e_{ir}(v; k, n)| \leq C \left(\frac{q_n}{n}\right)^{r-i}, \quad i=r, r+1, \dots, v-1,$$

where C is independent of n and k .

Proof. We prove this by induction on i . From $h_{rk}^{(r)}(x_k) = 1$ and (2.2), it follows that $e_{rr}(k) = 1/r!$. Thus, the case $i=r$ holds. By (2.2) and the fact $h_{rk}^{(i)}(x_k) = 0$, $r+1 \leq i \leq v-1$, we easily see

$$e_{ir}(k) = - \sum_{s=r}^{i-1} \frac{1}{(i-s)!} e_{sr}(k) (l_k^v)^{(i-s)}(x_k), \quad r+1 \leq i \leq v-1,$$

Since $M_n(x_k) \leq Cq_n^{m-1}$ by (I), (i), it follows from Lemma 2 that $|(l_k^v)^{(s)}(x_k)| \leq Cq_n^{s(m-1)} \leq C(q_n/n)^{-s}$ for every s , where C is independent of n and k . This inequality and the assumption of induction lead to

$$\begin{aligned}|e_{ir}(k)| &\leq C \sum_{s=r}^{i-1} |e_{sr}(k)| |(l_k^v)^{(i-s)}(x_k)| \\ &\leq C \sum_{s=r}^{i-1} \left(\frac{q_n}{n}\right)^{r-s} \left(\frac{q_n}{n}\right)^{-(i-s)} \leq C \left(\frac{q_n}{n}\right)^{r-i},\end{aligned}$$

where C is independent of n and k .

Q.E.D.

We continue the proof of Proposition. We first estimate \sum_1 . We may assume $J \neq \emptyset$. Then, by (I), (i), $J = \{k(x)\}$, where $k(x)$ is the number satisfying $x_{k(x)n} = x_{(x,n)}$. The number $k(x)$ may depend on n . Since $|x - x_{k(x)n}| \leq \delta q_n/n$, it follows from Lemma 3 and the mean value theorem that $\sum_1 = R_{k(x)}(i, r, n; x) \leq C |p'_n(\xi)/p'_n(x_{k(x)n})|^v \cdot (q_n/n)^r$, where C is independent of n and x , and ξ is between x and $x_{k(x)n}$. Since $\kappa < \tilde{\kappa}$ and $x \in [-\kappa q_n, \kappa q_n]$, it follows that $x_{k(x)n} \in [-\tilde{\kappa} q_{n-1}, \tilde{\kappa} q_{n-1}]$ for $n \geq n_0$, where n_0 is a number depending only on $\kappa, \tilde{\kappa}$ and m . By Lemma 1, we have $|p'_n(\xi)/p'_n(x_{k(x)n})| \leq C$ for $n \geq n_0$ since $|\xi - x_{k(x)n}| \leq |x - x_{k(x)n}| \leq \delta q_n/n \leq \tilde{\delta} q_n/n$. Therefore, we have

$$\sum_1 \leq C(q_n/n)^r \quad (2.3)$$

for $x \in [-\kappa q_n, \kappa q_n]$ and $n \geq n_0$, where C is independent of n and x .

We next treat \sum_2 . Let $1 \leq j \leq \lambda(n)$ and $k \in J(j)$. By Lemma 3, we have $R_k(i, r, n; x) \leq C j^i |p_n(x)/\{(x - x_k)p'_n(x_k)\}|^v (q_n/n)^r$ with C independent of n, x and j . We assume $\kappa < \min\{\kappa_1, \kappa_2\}$, where κ_1 and κ_2 are the constants in (I), (ii) and (II), respectively. By (I), (ii), we see that there exists a number n_1 such that if $n \geq n_1$, then $|x - x_{(x,n)}| \leq C_2 q_n/n$ for $x \in [-\kappa q_n, \kappa q_n]$, where n_1 depends only on κ, κ_1 and κ_2 . Thus, by (II) we have

$$|p_n(x)| \leq C w(x)^{-1} q_n^{-1/2}, \quad x \in [-\kappa q_n, \kappa q_n], n \geq n_1. \quad (2.4)$$

Since there exists a number n_2 depending only on $\kappa, \tilde{\kappa}$ and m such that $[-\kappa q_n, \kappa q_n] \subset [-\tilde{\kappa} q_{n-1}, \tilde{\kappa} q_{n-1}]$ for $n \geq n_2$, it follows from Lemma 1 that $|(x - x_k)p'_n(x_k)|^{-1} \leq C j^{-1} q_n^{1/2} w(x_k)$ for $k \in J(j)$ and $n \geq n_2$. Thus, $R_k(i, r, n; x) \leq C \{w(x_k)/w(x)\}^v j^{i-v} (q_n/n)^r \leq C w(x)^{-v} j^{-1} \cdot (q_n/n)^r$ for $x \in [-\kappa q_n, \kappa q_n]$ and $n \geq \max\{n_1, n_2\}$. Since every $J(j)$ has at most two elements, it follows that

$$\sum_2 \leq C \left(\frac{q_n}{n}\right)^r w(x)^{-v} \sum_{j=1}^{\lambda(n)} j^{-1} \leq C w(x)^{-v} \left(\frac{q_n}{n}\right)^r \log n \quad (2.5)$$

for $x \in [-\kappa q_n, \kappa q_n]$ and $n \geq \max\{n_1, n_2\}$, where C is independent of n and x .

Lastly, we estimate \sum_3 . Let $k \in I$ and $x \in [-\kappa q_n, \kappa q_n]$. Since $|x - x_k| \leq (\kappa + K_1) q_n$ for every k , it follows from Lemma 3 that $R_k(i, r, n; x) \leq C |p_n(x)/\{(x - x_k)p'_n(x_k)\}|^v (q_n/n)^{r-i} q_n^i$ with C independent of x and n . By (2.4) and $|x - x_k| \geq \delta q_n/n$, we have $R_k(i, r, n; x) \leq C w(x)^{-v} n^{v-r+i} q_n^{r-3v/2} |p'_n(x_k)|^{-v}$ and thus,

$$\sum_3 \leq C w(x)^{-v} n^{v-r+i} q_n^{r-3v/2} \sum_{k \in I} |p'_n(x_k)|^{-v}$$

for $x \in [-\kappa q_n, \kappa q_n]$, where C is independent of n and x . The sum $\sum_{k \in I} |p'_n(x_k)|^{-v}$ is treated by the following lemma.

Let λ_{kn} , $k=1, 2, \dots$ be the Cotes numbers which appear in the Gauss-Jacobi quadrature formula

$$\sum_{k=1}^n p(x_{kn}) \lambda_{kn} = \int_{-\infty}^{\infty} p(x) w^2(x) dx$$

valid for all polynomials $p(x)$ of degree at most $2n-1$ (cf. [11]).

LEMMA 4 [5, Lemma 7]. *Let $\tau > 0$. Then,*

$$\sum_{k: |x_k| \geq \tau q_n} p'_n(x_k)^{-2} \leq C q_n^{-2m+3} w^2(\tau q_n),$$

where C is independent of n .

By the lemma and $v/2 \geq 1$, we have

$$\begin{aligned} \sum_{k \in I} |p'_n(x_k)|^{-v} &\leq \left\{ \sum_{k \in I} |p'_n(x_k)|^{-2} \right\}^{v/2} n^{1/2} \\ &\leq C \{q_n^{-2m+3} w^2(\kappa q_n)\}^{v/2} n^{1/2} = C q_n^{(-2m+3)v/2} e^{-\mu n} n^{1/2}, \end{aligned}$$

where C is independent of n and x , and $\mu = v\kappa^m m^{-1}$. Therefore, we have

$$\Sigma_3 \leq C w(x)^{-v} \left(\frac{q_n}{n} \right)^r e^{-\mu n} n^{i+1/2} \quad (2.6)$$

for $x \in [-\kappa q_n, \kappa q_n]$, where C is independent of n and x . The proof of Proposition is concluded by combining (2.3), (2.5) and (2.6).

We remark that by a more refined estimate on Σ_2 we can get $\sum_{k=1}^n |h_{rkn}(v; x)| \leq C \{x^{m-1} w(x)^{-v} + \log n\} (q_n/n)^r$ for $x \in [-\kappa q_n, \kappa q_n]$. For the sake of brevity, we omit details.

The fundamental estimate (1.1) established now allows us to prove the Theorem. Let N be a non-negative integer, and let λ and μ be constants such that $0 < \lambda < \mu$. Let $P^*(x)$ be the polynomial of best approximation of order $n-1$ to $f \in C^N(\mathbf{R})$ on the interval $[-\mu q_n, \mu q_n]$. We put $g(t) = f(\mu q_n t)$ and $R^*(t) = P^*(\mu q_n t)$. Then, we note that $R^*(t)$ is the polynomial of best approximation of order $n-1$ to $g(t)$ on the interval $[-1, 1]$. Applying (IV), (i) and changing variable $x = \mu q_n t$, we have

$$\begin{aligned}
|g^{(j)}(t) - R^{*(j)}(t)| &= (\mu q_n)^j |f^{(j)}(x) - P^{*(j)}(x)| \\
&\leq C_6 (n-1)^{-N} \left\{ \Delta_{n-1} \left(\frac{x}{\mu q_n} \right) \right\}^{-j} \\
&\quad \times (\mu q_n)^N E_{n-1-N}(f^{(N)}; [-\mu q_n, \mu q_n]) \\
&\leq C (n-1)^{-N} \left\{ \Delta_{n-1} \left(\frac{x}{\mu q_n} \right) \right\}^{-j} \\
&\quad \times (\mu q_n)^N \omega \left([-\mu q_n, \mu q_n]; f^{(N)}; \frac{q_n}{n} \right)
\end{aligned}$$

for $j=0, 1, \dots, N$, $n-1 > N$ and $|x| \leq \mu q_n$, where C is a constant depending only on μ and N . Here, we used Jackson's theorem (cf. [12, 5.1(1)]) and the fact $E_{n-1-N}(g^{(N)}; [-1, 1]) = (\mu q_n)^N \cdot E_{n-1-N}(f^{(N)}; [-\mu q_n, \mu q_n])$. For $|x| \leq \lambda q_n$, we have $\Delta_{n-1}(x/(\mu q_n)) \geq (n-1)^{-1} \{1 - (\lambda/\mu)^2\}^{1/2}$. Thus, if $0 < \lambda < \mu$, then for $|x| \leq \lambda q_n$,

$$|f^{(j)}(x) - P^{*(j)}(x)| \leq C \left(\frac{q_n}{n} \right)^{N-j} \omega \left([-\mu q_n, \mu q_n]; f^{(N)}; \frac{q_n}{n} \right), \quad (2.7)$$

for $j=0, 1, \dots, N$ and $n-1 > N$, where C is a constant depending only on N , λ and μ .

Let $\delta(x) = |x| + 1$ for $|x| \leq 1$ and $\delta(x) = |x| + |x|^{1+m}$ for $|x| > 1$. The function $\delta(x)$ has first been introduced by [1] for the case $m=2$. Let $x \in \mathbf{R}$ be fixed. We apply (III) to the polynomial $R(t) = L_n(\delta(x)t) - P^*(\delta(x)t)$ of degree at most $vn-1$. Then, we have $|R^{(j)}(t)| \leq C_5 \Delta_{vn-1}(t)^{-j} \max_{|s| \leq 1} |R(s)|$ for $j=0, 1, \dots$ and $|t| \leq 1$. We use this inequality for $t = x/\delta(x)$. Then, we easily see that for $j=0, 1, \dots$,

$$\begin{aligned}
|L_n^{(j)}(x) - P^{*(j)}(x)| &\leq C_5 \left\{ \delta(x) \Delta_{vn-1} \left(\frac{x}{\delta(x)} \right) \right\}^{-j} \max_{|u| \leq \delta(x)} |L_n(u) - P^*(u)| \\
&\leq C \frac{n^j}{(\delta(x)^2 - x^2)^{j/2}} \max_{|u| \leq \delta(x)} |L_n(u) - P^*(u)| \\
&\leq C(1 + |x|^{j(m-2)/2}) n^j \max_{|u| \leq \delta(x)} |L_n(u) - P^*(u)|, \quad (2.8)
\end{aligned}$$

where C is a constant depending only on j and v . Let K be a constant such that $K_1 < K$, and let c be a constant such that $0 < c < \min\{K_1, \kappa\}$, where

K_1 and κ are the constants in (I), (i) and Proposition, respectively. By (2.7) and (2.8), we have, for $|x| \leq cq_n$ and $j = 0, 1, \dots, N$,

$$\begin{aligned} & |L_n^{(j)}(x) - f^{(j)}(x)| \\ & \leq |L_n^{(j)}(x) - P^{*(j)}(x)| + |P^{*(j)}(x) - f^{(j)}(x)| \\ & \leq C \left\{ (1 + |x|^{j(m-2)/2}) n^j \max_{|u| \leq \delta(x)} |L_n(u) - P^*(u)| \right. \\ & \quad \left. + \left(\frac{q_n}{n}\right)^{N-j} \omega([-Kq_n, Kq_n]; f^{(N)}; \frac{q_n}{n}) \right\}, \end{aligned} \quad (2.9)$$

where C is a constant independent of n, x and f .

It is enough to estimate $|L_n(u) - P^*(u)|$ for $|u| \leq \delta(x)$. Since the degree of $P^*(u)$ does not exceed $m-1$, it follows that $L_n(v-1, v; P^*, u) = P^*(u)$, which leads to

$$\begin{aligned} L_n(u) - P^*(u) &= L_n(u) - L_n(v-1, v; P^*, u) \\ &= \sum_{k=1}^n \sum_{r=0}^l \{f^{(r)}(x_k) - P^{*(r)}(x_k)\} h_{rk}(u) \\ &\quad - \sum_{k=1}^n \sum_{r=l+1}^{v-1} P^{*(r)}(x_k) h_{rk}(u). \end{aligned} \quad (2.10)$$

We note that if $v-1 = l$, then the second sum vanishes. Let N be an integer such that $l \leq N$. Since $|x_k| \leq K_1 q_n$ for all k by (I), (i), it follows from (2.7) that for $k = 1, 2, \dots, n$ and $r = 0, 1, \dots, l$,

$$\begin{aligned} & |f^{(r)}(x_k) - P^{*(r)}(x_k)| \\ & \leq C \left(\frac{q_n}{n}\right)^{N-r} \omega\left([-Kq_n, Kq_n]; f^{(N)}; \frac{q_n}{n}\right), \end{aligned} \quad (2.11)$$

where C is a positive constant depending only on N, K_1 and K . If $r > l$, then by (IV), (ii) and by changing variables,

$$|P^{*(r)}(x_k)| \leq C \left(\frac{q_n}{n}\right)^{l-r} \omega\left([-Kq_n, Kq_n]; f^{(l)}; \frac{q_n}{n}\right), \quad (2.12)$$

for $k = 1, 2, \dots, n$, where C is a positive constant depending only on r, K_1 and K . Applying the estimates (2.11) and (2.12) to the expression (2.10), we have

$$|L_n(u) - P^*(u)| \leq \begin{cases} C\omega_N\left(\frac{q_n}{n}\right) \sum_{r=0}^{v-1} \left(\frac{q_n}{n}\right)^{N-r} \sum_{k=1}^n |h_{rk}(u)| & (v-1=l), \\ C\omega_l\left(\frac{q_n}{n}\right) \sum_{r=0}^{v-1} \left(\frac{q_n}{n}\right)^{l-r} \sum_{k=1}^n |h_{rk}(u)| & (v-1>l), \end{cases} \quad (2.13)$$

where C is a constant independent of n, x and f , and $\omega_j(q_n/n)$ stands for $\omega([Kq_n, Kq_n]; f^{(j)}; q_n/n)$. Note that there exists a number n_0 such that if $n \geq n_0$ then $\delta(x) \leq \kappa q_n$ for x with $|x| \leq cq_n$. Then, by Proposition and the inequality $\max_{|u| \leq \delta(x)} e^{v|u|^m/2} \leq Ce^{v\kappa^m/2}$ with C independent of x , we completes the proof of Theorem.

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